EVALUATION OF DEDEKIND SUMS, EISENSTEIN COCYCLES, AND SPECIAL VALUES OF L-FUNCTIONS

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Abstract. We define certain higher-dimensional Dedekind sums that generalize the classical Dedekind-Rademacher sums, and show how to compute them effectively using a generalization of the continued-fraction algorithm.

We present two applications. First, we show how to express special values of partial zeta functions associated to totally real number fields in terms of these sums via the Eisenstein cocycle introduced by the second author. Hence we obtain a polynomial-time algorithm for computing these special values. Second, we show how to use our techniques to compute certain special values of the Witten zeta-function, and compute some explicit examples.

1. Introduction

1.1. Let σ be a square matrix with integral columns $\sigma_j \in \mathbb{Z}^n$ ($j = 1, \ldots, n$), and let $L \subset \mathbb{Z}^n$ be a lattice of rank ≥ 1. Let $v \in \mathbb{R}^n$, and let $e \in \mathbb{Z}^n$ with $e_j \geq 1$. Associated to the data (L, σ, e, v) is the *Dedekind sum*

$$S = S(L, \sigma, e, v) := {\sum_{x \in L}}' \mathbf{e}(\langle x, v\rangle) \frac{\det \sigma}{\langle x, \sigma_1\rangle^{e_1} \cdots \langle x, \sigma_n\rangle^{e_n}}. \tag{1}$$

Here $\langle x, y\rangle := \sum x_i y_i$ is the usual scalar product on $\mathbb{R}^n$, $\mathbf{e}(t)$ is the character $\exp(2\pi i t)$, and the prime next to the summation means to omit terms for which the denominator vanishes. This series converges absolutely if all $e_j > 1$, but only conditionally if $e_j = 1$ for some j. In the latter case, we define the value of S by the Q-limit

$$\left.{\sum_{x \in L}}' a(x)\right|_Q := \lim_{t \to \infty} \Big({\sum_{\substack{x \in L \\ |Q(x)| < t}}}' a(x) \Big), \tag{2}$$

where Q is any finite product of real-valued linear forms on $\mathbb{R}^n$ that do not vanish on $\mathbb{Q}^n \smallsetminus \{0\}$. As we explain in §2.6, this limit depends on Q in a rather simple way.

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Nevertheless, to keep notation to a minimum, we assume for now that the series (1) converges absolutely.

1.2. The arithmetic nature of the values of S is well known. Up to a power of $2\pi i$, they are always rational numbers if $v \in \mathbb{Q}^n$. However, the explicit calculation of these values is not easy, especially if $|\det \sigma|$ is large. In this paper, we exhibit a polynomial-time algorithm for calculating S efficiently. Before stating our main result, we briefly review a few special cases.

In the case $n = 1$, $\sigma = 1$, $L = \mathbb{Z}$, we have

$$S(L, \sigma, e, v) = -\frac{(2\pi i)^e}{e!} \mathscr{B}_e(v), \tag{3}$$

where $\mathscr{B}_e(v)$ is the periodic function (with period lattice $\mathbb{Z}$) that coincides with the classical Bernoulli polynomial $B_e(v)$ on the interval $0 < v < 1$.

More generally, if $L = \mathbb{Z}^n$, then we show in Proposition 3.11 that

$$S(L, \sigma, e, v) = \kappa \sum_{r \in L/\sigma L} \mathscr{B}_{e_1}(u_1) \cdots \mathscr{B}_{e_n}(u_n), \tag{4}$$

where

$$u = \sigma^{-1}(r + v) \quad \text{and} \quad \kappa = \operatorname{sgn}(\det \sigma) \prod_{j=1}^{n} \frac{(-2\pi i)^{e_j}}{e_j!}. \tag{5}$$

The finite sum on the right of (4) is a classical Dedekind sum. Although theoretically satisfying, from a computational point of view this sum is of interest only if $|\det \sigma|$ (the number of summands) is relatively small.

1.3. In general, if $\operatorname{rank} L < n$, no simple analogue of (4) seems to exist except for the special case where $\operatorname{rank} L$ equals the number of linear forms $\langle x, \sigma_j \rangle$ in (1) that are not proportional to each other when restricted to the subspace $L \otimes \mathbb{R}$. Dedekind sums of this special type are called *diagonal*. It is clear that the finite sum formula (4) remains valid (modulo obvious modifications) for all diagonal Dedekind sums.

Let us further say that a diagonal Dedekind sum is *unimodular* if the corresponding finite sum has exactly one term. As a measure of the deviation from unimodularity, we introduce an integer-valued function $\| \ \|$, called the *index*, on the set of all Dedekind sums (Definition 3.6). For example, if $L = \mathbb{Z}^n$, then $\|S\| = |\det \sigma|$. In particular, S is said to be unimodular if and only if $\|S\| = 1$.

1.4. We are now ready to describe our main result.

1.5. Theorem. *Every Dedekind sum $S(L, \sigma, e, v)$ can be expressed as a finite rational linear combination of unimodular diagonal sums. If n, $\operatorname{rank} L$, and e are fixed, then this expression can be computed in time polynomial in $\log \|S\|$. Moreover, the number of terms in this expression is bounded by a polynomial in $\log \|S\|$.*

The special case $n = 2$ and $e = (1, 1)$ has been known for a very long time. This is the case of the classical Dedekind-Rademacher sums, where the simple form of their reciprocity law combined with the Euclidean algorithm immediately yields a polynomial-time algorithm for computing their values (cf. Example 7.5). Our proof of Theorem 1.5 is similar. First we prove a general reciprocity law for the sums S. Applying and specializing this law repeatedly yields a representation of S as a linear combination of diagonal sums. Combining it further with the algorithm of Ash-Rudolph [1] finally yields a representation by unimodular diagonal sums.

1.6. Among the many applications, we review in this paper the problem of calculating the Eisenstein cocycle [7] on $GL_n(\mathbb{Q})$ that, when combined with Theorem 1.5, yields a polynomial-time algorithm for calculating special values of partial zeta functions of totally real number fields (§§6–7). These special values are of great interest in view of the many conjectures they are subject to (Leopoldt conjecture, Brumer-Stark conjecture, etc.). We also review the connection between Witten's zeta function and Dedekind sums, and use our techniques to give some explicit formulas for special values of these functions (§8).

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2. The modular symbol algorithm and Dedekind reciprocity

2.1. Let $\sigma_1, \dots, \sigma_n \in \mathbb{Z}^n$ be nonzero primitive points, and let $D = |\det(\sigma_1, \dots, \sigma_n)|$. Let $w \in \mathbb{Z}^n$ be another nonzero primitive point, and define

$$D_i(w) := |\det(\sigma_1, \dots, \widehat{\sigma}_i, \dots, \sigma_n, w)|, \quad i = 1, \dots, n.$$

The following basic result will play a key role:

2.2. Proposition. [1, 2] *If $D > 1$, then there exists $w \in \mathbb{Z}^n \smallsetminus \{0\}$ such that*

$$0 \le D_i(w) < D^{(n-1)/n}, \quad i = 1, \dots, n, \tag{6}$$

and at least one $D_i(w) \neq 0$. Moreover, for fixed n the point w can be constructed in polynomial time.

Proof. (Sketch) Here we only show that w exists. Let P be the open parallelotope

$$P := \Big\{ \sum \lambda_i \sigma_i \;\Big|\; |\lambda_i| < D^{-1/n} \Big\}.$$

Then P is an n-dimensional centrally symmetric convex body with volume 2^n. By Minkowski's theorem (cf. [3, IV.2.6]), $P \cap \mathbb{Z}^n$ contains a nonzero point. This is the desired point w.

□

2.3. *Remark.* Ash-Rudolph [1] show that w satisfies $0 \le D_i(w) < D$. They also show how to construct w using the Euclidean algorithm. The stronger estimate (6) and the statement about polynomial time are due to Barvinok [2, Lemma 5.2]. In practice, to efficiently construct w such that $D_i(w)$ is small, one may use LLL-reduction [5, §3].

2.4. We now state the Dedekind reciprocity law. For any nonzero point $v \in \mathbb{R}^n$, let $v^\perp$ be the hyperplane $\{x \mid \langle v, x\rangle = 0\}$. Let Q be a finite product of real-valued linear forms on $\mathbb{R}^n$ that do not vanish on $\mathbb{Q}^n \smallsetminus \{0\}$ and recall that the notation

$$\sum_{x\in L}{}' a(x)\Big|_Q$$

means the sum is to be evaluated using the Q-limit (§1.1, §2.6).

2.5. Proposition. *Let $\sigma_0, \dots, \sigma_n \in \mathbb{Z}^n$ be nonzero. For $j = 0, \dots, n$, let σ^j be the matrix with columns $\sigma_0, \dots, \widehat{\sigma}_j, \dots, \sigma_n$. Fix $L \subseteq \mathbb{Z}^n$, and assume $e = \mathbf{1} := (1, \dots, 1)$. Then for any $v \in \mathbb{R}^n$, we have the following identity among Dedekind sums:*

$$\sum_{j=0}^{n} (-1)^j S(L, \sigma^j, \mathbf{1})\Big|_Q = \sum_{j=0}^{n} (-1)^j S(L \cap \sigma_j^\perp, \sigma^j, \mathbf{1})\Big|_Q. \tag{7}$$

Proof. Let $D_j = \det \sigma^j$. We have an identity for rational functions of x

$$\sum_{j=0}^{n} \frac{(-1)^j D_j}{\prod_{k\neq j}\langle x, \sigma_k\rangle} = 0, \tag{8}$$

valid for any $x \in \mathbb{R}^n$ satisfying $\langle x, \sigma_j\rangle \neq 0$ for $j = 0, \dots, n$. To see this, consider the $(n + 1) \times (n + 1)$ matrix

$$\begin{pmatrix} \langle x, \sigma_0\rangle & \dots & \langle x, \sigma_n\rangle \\ \sigma_0 & \dots & \sigma_n \end{pmatrix}. \tag{9}$$

This matrix is singular, since the first row is a linear combination of the others. Expanding by minors along the top row, and dividing by $\prod\langle x, \sigma_k\rangle$ yields (8).

To pass from (8) to (7), we need to incorporate the exponential character and sum over L using Q. There is no obstruction to doing this, although we must omit terms where *any* linear form vanishes. We obtain the expression

$$\sum_{j=0}^{n} (-1)^j \sum \mathbf{e}(\langle x, v\rangle) \frac{D_j}{\prod_{k\neq j}\langle x, \sigma_k\rangle}\Big|_Q = 0, \tag{10}$$

where the inner sum is taken over all $x \in L$ with $\langle x, \sigma_k\rangle \neq 0$ for $k = 0, \dots, n$.

In (10), the jth inner sum corresponds with the Dedekind sum $S(L, \sigma^j, \mathbf{1})$ except for the terms with $\langle x, \sigma_j\rangle = 0$ and $\langle x, \sigma_k\rangle \neq 0$ for $k \neq j$. In other words, to make the

jth sum into a Dedekind sum, we must add

$$\sum_{x \in L \cap \sigma_j^{\perp}}' \mathbf{e}(\langle x, v\rangle) \frac{D_j}{\prod_{j \neq k} \langle x, \sigma_k \rangle}\bigg|_Q . \tag{11}$$

Simultaneously adding and subtracting (11) to (8) yields (7). $\square$

2.6. We now recall the Q-limit formula from [7, Theorem 2]. Let

$$Q(y) = \prod_{i=1}^{m} Q_i(y)$$

be a product of $m \geq 1$ linear forms

$$Q_i(y) = \sum_{j=1}^{n} Q_{ij} y_j,$$

with rationally independent real coefficients Q_{ij}. We think of Q as an $m \times n$ matrix with rows Q_i.

Given a vector $e = (e_1, \ldots, e_n)$ of positive integers and a vector $v \in \mathbb{R}^n$, let

$$J = \{ j \mid e_j = 1 \text{ and } v_j \in \mathbb{Z} \}.$$

If $\# J \equiv 0 \mod 2$, define

$$\mathbb{B}_e(v, Q) = \frac{1}{m} \sum_{i=1}^{m} \left(\prod_{j \in J} \frac{\operatorname{sgn} Q_{ij}}{2} \right) \prod_{j \notin J} \mathscr{B}_{e_j}(v_j), \tag{12}$$

otherwise let $\mathbb{B}_e(v, Q) = 0$. In particular, if $J = \varnothing$, then

$$\mathbb{B}_e(v, Q) = \prod_{j=1}^{n} \mathscr{B}_{e_j}(v_j).$$

Now the Q-limit formula can be stated as follows. Let Id_n be the $n \times n$ identity matrix. Then

$$S(\mathbb{Z}^n, \mathrm{Id}_n, e, v)\Big|_Q = \kappa \, \mathbb{B}_e(v, Q), \tag{13}$$

where

$$\kappa = \prod_{j=1}^{n} \frac{(-2\pi i)^{e_j}}{e_j!}. \tag{14}$$

3. Diagonality and unimodularity

3.1. We begin with some simplifying assumptions to ease the exposition. We define the *rank* of $S = S(L, \sigma, e, v)$ to be the rank of the lattice L.

Suppose that the rank of S is ℓ, and for any k let $Z^k \subseteq \mathbb{Z}^n$ be the sublattice spanned by the first k standard basis vectors. We claim that S can be computed using Z^ℓ instead of L. Indeed, writing $L = gZ^\ell$ with a matrix $g \in GL_n(\mathbb{Q})$, and letting $\sigma' = g^t\sigma$, $v' = g^t v$, and $Q' = Qg$, we have

$$S(L, \sigma, e, v)\Big|_Q = (\det g)^{-1} S(Z^\ell, \sigma', e, v')\Big|_{Q'}.$$

The entries of σ' need not be integral, but after multiplying by an appropriate rational factor we can assume this is true. In fact, by further multiplication of σ' by rational numbers and permuting columns, we can write

$$S(L, \sigma, e, v)\Big|_Q = qS(Z^\ell, \sigma', e, v')\Big|_{Q'}, \quad q \in \mathbb{Q}^\times,$$

where the pair (Z^ℓ, σ') satisfies the following:

(i) For each column σ'_j, the vector $\sigma'_j \cap Z^\ell$ is primitive and integral.
(ii) If two columns of σ' induce proportional linear forms on Z^ℓ, then these two linear forms coincide on Z^ℓ, and are adjacent columns of σ'.

3.2. Definition. We say that a rank ℓ Dedekind sum is *normalized* if the conditions above are met.

3.3. Let $S(Z^\ell, \sigma, e, v)$ be a normalized Dedekind sum, and let $N = \sum e_j$. We claim that without loss of generality, we need only consider sums for which $e = \mathbf{1}$. Indeed, let $R^\ell \to R^n \to \mathbb{R}^N$ be the spans of the first ℓ (respectively n) basis vectors in $\mathbb{R}^N$, and identify $\mathbb{R}^n$ with R^n. Let Q' be a product of linear forms on $\mathbb{R}^N$ such that Q' restricted to R^n equals Q restricted to $\mathbb{R}^n$. We claim that we can construct an $N \times N$ matrix σ' and $v' \in \mathbb{R}^N$ such that

$$S(Z^\ell, \sigma, e, v)\Big|_Q = S(Z^\ell, \sigma', \mathbf{1}, v')\Big|_{Q'}. \tag{15}$$

To see this, let $\pi\colon \mathbb{R}^N \to \mathbb{R}^\ell$ be the projection on the first ℓ components. Given $v \in R^n$, call any $\tilde{v} \in \mathbb{R}^N$ such that $\pi(\tilde{v}) = \pi(v)$ a *lift* of v.

Now to construct $S(Z^\ell, \sigma', \mathbf{1}, v')$, let v' be any lift of v. If the restriction of the linear form σ_j to Z^ℓ appears on the left of (15) with multiplicity e_j, set j columns of σ' to be j different lifts of σ_j. If we choose these lifts so that $\det\sigma' = \det\sigma$, then we obtain (15).

3.4. Definition. We say that a Dedekind sum with $e = \mathbf{1}$ is *properly embedded.*

3.5. Let $S(Z^\ell, \sigma, \mathbf{1}, v)$ be a properly embedded, normalized Dedekind sum. Let $[\![n]\!]$ be the set $\{1, \dots, n\}$.

3.6. Definition. The *index* of S, denoted $\|S\|$, is defined to be

$$\operatorname*{Max}_{I\subset[\![n]\!]} \left|\det\bigl(\pi(\sigma_{i_1}), \dots, \pi(\sigma_{i_\ell})\bigr)\right|,$$

where the maximum is taken over all subsets $I = \{i_1, \dots, i_\ell\}$ of cardinality ℓ. A Dedekind sum is called *unimodular* if $\|S\| = 1$.

3.7. Now define a partition

$$[\![n]\!] = \bigsqcup_{k=1}^{s} I_k, \quad \ell \le s \le n \tag{16}$$

as follows. Put

$$i, j \in I_k \quad \text{if and only if} \quad \pi(\sigma_i) = \pi(\sigma_j).$$

In other words, two elements of $[\![n]\!]$ are in the same set of the partition if the corresponding columns of σ induce the same linear form on Z^ℓ.

Let $p_k = \#I_k$. The vector $p(S) = (p_1, \dots, p_s)$ is called the *type* of S. To emphasize the type, we relabel the columns of σ as

$$(\sigma_1^1, \dots, \sigma_1^{p_1}, \sigma_2^1, \dots, \sigma_2^{p_2}, \dots, \sigma_s^1, \dots, \sigma_s^{p_s}). \tag{17}$$

For any $k = 1, \dots, s$ and any $i = 1, \dots, p_k$, we denote the point $\pi(\sigma_k^j) \in \mathbb{Z}^\ell$ by σ_{I_k}.

3.8. Definition. A Dedekind sum is called *diagonal* if $p(S)$ has length ℓ.

We omit the proof of the following simple lemma:

3.9. Lemma. *Any normalized, properly embedded rank 1 Dedekind sum is both diagonal and unimodular.*

3.10. The virtue of diagonality and unimodularity is the following:

3.11. Proposition. *Keep the notation of the preceding section. Let $S(Z^\ell, \sigma, \mathbf{1}, v)$ be properly embedded, normalized, and diagonal. Let ρ be the $\ell \times \ell$ matrix $(\sigma_{I_1}, \dots, \sigma_{I_\ell})$, and let $(p_1, \dots, p_\ell)$ be the type of S. Then $S|_Q$ is well-defined. Moreover,*

$$S(Z^\ell, \sigma, \mathbf{1}, v)\Big|_Q = \frac{\kappa \det \sigma}{|\det \rho|} \sum_{r \in \mathbb{Z}^\ell/\rho\mathbb{Z}^\ell} \mathbb{B}_p(u, Q'), \tag{18}$$

where

$$u = \rho^{-1}(r + \pi(v)), \quad Q' = Q'' \circ \rho^{-t} \circ \pi, \quad \textit{and} \tag{19}$$

$$\kappa = \prod_{j=1}^{\ell} \frac{(-2\pi i)^{p_j}}{p_j!}. \tag{20}$$

Here Q'' is the restriction of Q to $\mathbb{R}^\ell$ into R^ℓ. If S is unimodular, the sum (18) *has only one term.*

Proof. The second statement follows easily from the first, so we focus on the first. By definition, we have

$$S(Z^\ell, \sigma, \mathbf{1}, v)\Big|_Q = \sum_{x\in Z^\ell}{}' \mathbf{e}(\langle x, v\rangle)\frac{\det\sigma}{\prod_{1\le k\le \ell}\langle x, \sigma_k^1\rangle^{p_k}}\Big|_Q. \tag{21}$$

Letting $Q' = Q''\circ\rho^{-t}\circ\pi$ and $v' = \rho^{-1}\pi(v)$, the right of (21) becomes

$$\det(\sigma)\sum_{y\in\rho^t\mathbb{Z}^\ell}{}' \mathbf{e}(\langle y, v'\rangle)\frac{\det\rho}{\prod_{1\le k\le\ell} y_k^{p_k}}\Big|_{Q'}. \tag{22}$$

Inserting the character relations

$$\sum_{r\in\mathbb{Z}^\ell/\rho\mathbb{Z}^\ell}\mathbf{e}(\langle y, \rho^{-1}r\rangle) = \begin{cases} 0, & y\in\mathbb{Z}^\ell\smallsetminus\rho^t\mathbb{Z}^\ell,\\ \#(\mathbb{Z}^\ell/\rho\mathbb{Z}^\ell), & y\in\rho^t\mathbb{Z}^\ell,\end{cases} \tag{23}$$

we obtain

$$S(Z^\ell, \sigma, \mathbf{1}, v) = \frac{\det\sigma}{|\det\rho|}\sum_{r\in\mathbb{Z}^\ell/\rho\mathbb{Z}^\ell}\ \sum_{y\in\mathbb{Z}^\ell}{}' \frac{\mathbf{e}(\langle y, \rho^{-1}(r+v)\rangle)}{\prod y_k^{p_k}}\Bigg|_{Q'} \tag{24}$$

$$= \frac{\det\sigma}{|\det\rho|}\sum_{r\in\mathbb{Z}^\ell/\rho\mathbb{Z}^\ell} S(\mathbb{Z}^\ell, \mathrm{Id}_n, p, u)\Big|_{Q'} \tag{25}$$

where $u = \rho^{-1}(r+\pi(v))$.

The proposition follows now from the Q-limit formula (13). ☐

4. Algorithms

4.1. In this section we prove that any Dedekind sum is a $\mathbb{Q}$-linear combination of diagonal, unimodular sums. We begin with a lemma. For simplicity, we abbreviate the Dedekind sum to $S(Z^\ell, \sigma)$.

4.2. Lemma. *Let $S(Z^\ell, \sigma)$ be a normalized, properly embedded Dedekind sum, and let σ_i be a column of σ. Then*

$$\big\|S(Z^\ell\cap\sigma_i^\perp, \sigma)\big\| \le \big\|S(Z^\ell, \sigma)\big\|.$$

Proof. Without loss of generality, assume that $\sigma_i = \sigma_1$. We can represent σ as

$$\sigma = \left(\begin{array}{ccc|ccc|c|cc} \sigma_{I_1} & \cdots & \sigma_{I_1} & \sigma_{I_2} & \cdots & \sigma_{I_2} & \cdots & \cdots & \sigma_{I_s}\\ * & \cdots & * & * & \cdots & * & \cdots & \cdots & * \end{array}\right), \tag{26}$$

where there are p_k columns of the form $(\sigma_{I_k}, *)^t$. The stars in the last $n-\ell$ rows represent numbers that are irrelevant, since they don't affect the value of the sum.

Let $\gamma \in GL_n(\mathbb{Q})$ be a matrix that carries $Z^\ell \cap \sigma_1^\perp$ onto $Z^{\ell-1}$. For $k = 1, \dots, \ell$ let $\sigma'_{I_k} = \gamma\sigma_k^j$, where σ_k^j is any lift of σ_{I_k}. Then $\bar{\gamma}\sigma$ has the form

$$\bar{\gamma}\sigma = \left(\begin{array}{ccc|ccc|c|cc} 0 & \cdots & 0 & \sigma'_{I_2} & \cdots & \sigma'_{I_2} & \cdots & \cdots & \sigma'_{I_s} \\ \varepsilon & \cdots & \varepsilon & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ * & \cdots & * & * & \cdots & * & \cdots & \cdots & * \end{array}\right). \tag{27}$$

Here the row blocks have sizes $\ell - 1$, 1, and $n - \ell$, and $\varepsilon = \pm 1$.

Now let d be the determinant of any $(\ell-1) \times (\ell-1)$ minor from the top $\ell - 1$ rows of (27). This determinant will be the same up to sign as the determinant of an $\ell \times \ell$ minor containing σ_1 from the top ℓ rows of (26). Hence $|d| \leq \|S\|$, and the proof is complete.

□

Now we come to our first main result.

4.3. Theorem. *Let $S = S(Z^\ell, \sigma)$ be a normalized, properly embedded Dedekind sum. Then we may write*

$$S = \sum_{\varrho \in R} q_\varrho S(Z^\ell, \varrho) + \sum_{\tau \in T} q_\tau S(Z^{\ell-1}, \tau), \tag{28}$$

where

1. *the sets R and T are finite,*
2. *$q_\varrho, q_\tau \in \mathbb{Q}$,*
3. *each $S(Z^\ell, \varrho)$ is diagonal, and*
4. *each of the Dedekind sums on the right of* (28) *has index $\leq \|S\|$.*

Proof. Write $\sigma = (\sigma_1^1, \dots, \sigma_1^{p_1}, \sigma_2^1, \dots, \sigma_2^{p_2}, \dots, \sigma_s^1, \dots, \sigma_s^{p_s})$ as in (17). Permuting columns if necessary, we may assume that $\sigma_{I_1}, \dots, \sigma_{I_\ell}$ are linearly independent in Z^ℓ. We will show that we can write an expression like (28) so that the rank ℓ sums on the right have type

$$(p_1, \dots, p_i - 1, \dots, p_\ell, \dots, p_s + 1),$$

for some $1 \leq i \leq \ell$. Iterating this construction proves that we can have the rank ℓ sums on the right of (28) diagonal.

We proceed as follows. Since $\sigma_{I_1}, \dots, \sigma_{I_\ell}$ are linearly independent, we can find unique rational numbers α_j such that

$$\sigma_{I_s} = \sum_{j=1}^{\ell} \alpha_j \sigma_{I_j}. \tag{29}$$

Now we want to apply the relation in Proposition 2.5. For each σ_{I_j}, we choose p_j rational lifts $\tilde{\sigma}_j^1$, … ,$\tilde{\sigma}_j^{p_j}$, not necessarily equal to the columns of σ, and use them to form a matrix $\tilde{\sigma}$. Clearly $S(Z^\ell, \tilde{\sigma}) = S(Z^\ell, \sigma)$.

Now define

$$\tilde{w} = \sum_{j=1}^{\ell} \alpha_j \tilde{\sigma}_j^1.$$

Clearly $\pi(\tilde{w}) = \sigma_{I_s}$. Write $\tilde{\sigma}_i^j(\tilde{w})$ for the matrix made from σ by replacing the column σ_i^j with $\tilde{w}$. Using the columns of $\tilde{\sigma}$ and $\tilde{w}$ in (7), we find

$$S(Z^\ell, \tilde{\sigma}) = S(Z^\ell \cap \tilde{w}^\perp, \tilde{\sigma}) + \sum (\varepsilon_i^j) S\big(Z^\ell, \tilde{\sigma}_i^j(\tilde{w})\big) - \sum (\varepsilon_i^j) S\big(Z^\ell \cap (\tilde{\sigma}_i^j)^\perp, \tilde{\sigma}_i^j(\tilde{w})\big), \tag{30}$$

where $\varepsilon_i^j \in \{\pm 1\}$ is determined by the reciprocity law, and the sums are over pairs (i, j) satisfying $1 \leq i \leq s$ and $1 \leq j \leq p_i$.

We claim that the sums in (30) actually have only ℓ terms. This follows since the points

$$\tilde{\sigma}_1^1, \dots, \tilde{\sigma}_\ell^1, \tilde{w}$$

are dependent. Hence any sum such that $\tilde{\sigma}_i^j(\tilde{w})$ contains these points vanishes. Moreover, the sum $S(Z^\ell \cap \tilde{w}^\perp, \tilde{\sigma})$ is zero, since by construction a column of $\tilde{\sigma}$ induces a linear form vanishing on $Z^\ell \cap \tilde{w}^\perp$. Hence (30) becomes

$$S(Z^\ell, \tilde{\sigma}) = \sum_{i=1}^{\ell} S\big(Z^\ell, \tilde{\sigma}_i^1(\tilde{w})\big) - \sum_{i=1}^{\ell} S\big(Z^\ell \cap (\tilde{\sigma}_i^1)^\perp, \tilde{\sigma}_i^1(\tilde{w})\big). \tag{31}$$

Now consider the types of the rank ℓ Dedekind sums on the right of (31). If the type of S was

$$(p_1, \dots, p_i, \dots, p_\ell, \dots, p_s),$$

then the type of $S(\mathbb{Z}^\ell, \tilde{\sigma}_i^1(\tilde{w}))$ is

$$(p_1, \dots, p_i - 1, \dots, p_\ell, \dots, p_s + 1).$$

Hence by induction we can write S as a finite $\mathbb{Q}$-linear combination of diagonal rank ℓ Dedekind sums plus sums of lower rank, which proves 1–3 of the statement.

To complete the proof, we must show that the indices of the Dedekind sums on the right of (28) are no larger than $\|S\|$. Indeed, Lemma 4.2 implies that the indices on the right of (30) are no larger than $\|S\|$, and so the claim follows. $\square$

4.4. *Remark.* There are some simplifications in (31) that are worth mentioning if one wishes to implement the diagonalization algorithm. First, if we construct $\tilde{\sigma}$ so that $\det \tilde{\sigma} = 1$, then $\det \tilde{\sigma}_i^1 = \alpha_j$. This means that these determinants can be computed when one computes σ_{I_s}. Second, not all the terms on the right of (31) necessarily appear. In particular, the rank $\ell - 1$ sum $S\big(Z^\ell \cap (\tilde{\sigma}_i^1)^\perp, \tilde{\sigma}_i^1(\tilde{w})\big)$ appears on the right of (31) only if $p_i = 1$.

4.5. Theorem. *With the notation as in Theorem 4.3, we can write*

$$S = \sum_{\varrho\in R} q_\varrho S(Z^\ell, \varrho) + \sum_{\tau\in T} q_\tau S(Z^{\ell-1}, \tau), \tag{32}$$

where the sums of rank ℓ on the right are diagonal and unimodular, and the rank $\ell-1$ sums have index $\leq \|S\|$.

Proof. By Theorem 4.3, we may take S to be diagonal. Suppose that

$$D = |\det(\sigma_{I_1}, \dots, \sigma_{I_\ell})| > 1.$$

Then by Proposition 2.2, there exists $w \in Z^\ell$ such that

$$D_j = |\det(\sigma_{I_1}, \dots, \widehat{\sigma}_{I_j}, \dots, \sigma_{I_\ell}, w)|$$

satisfies $0 \leq D_j < D^{(\ell-1)/\ell}$. As in the proof of Theorem 4.3, write $w = \sum_{j=1}^{\ell} \alpha_j \sigma_{I_j}$, and set $\tilde{w} = \sum_{j=1}^{\ell} \alpha_j \sigma_j^1$.

Now apply Proposition 2.5, using $\tilde{w}$ and the columns of σ, to write S as a finite $\mathbb{Q}$-linear combination of new Dedekind sums. These sums won't be diagonal, but we can apply the proof of Theorem 4.3 with w playing the role of σ_{I_s}. The resulting sums will include lifts of w in their columns and will be diagonal. Thus the resulting rank ℓ sums will have index $< \|S\|$, and the sums of lower rank will satisfy the conditions in the statement of Theorem 4.3. By induction on the index, this completes the proof. ∎

4.6. Corollary. *Any Dedekind sum can be written as a finite $\mathbb{Q}$-linear combination of diagonal, unimodular sums.*

Proof. First normalize and embed properly. By Lemma 3.9, any rank 1 Dedekind sum is automatically unimodular and diagonal. The result follows by applying Theorems 4.3 and 4.5 and descending induction on the rank. ∎

5. Complexity

5.1. In this section we discuss the computational complexity of Corollary 4.6. In particular, we show that if n and ℓ are fixed, and $S = S(Z^\ell, \sigma)$ is normalized and properly embedded, then we can form a finite $\mathbb{Q}$-linear combination of diagonal, unimodular Dedekind sums

$$S = \sum_{\substack{\varrho\in R\\ k\leq\ell}} q_\varrho S(Z^k, \varrho),$$

where $\#R$ is bounded by a polynomial in $\log \|S\|$. As a corollary we obtain that this expression can be computed in polynomial time.

To do this, we must make a more detailed analysis of proofs in §4. We begin by analyzing diagonality.

5.2. Lemma. *Let $S = S(Z^\ell, \sigma)$ be a normalized, properly embedded Dedekind sum, and write*

$$S = \sum_{\varrho \in R} q_\varrho S(Z^\ell, \varrho) + \sum_{t \in T} q_\tau S(Z^{\ell-1}, \tau) \tag{33}$$

as in Theorem 4.3, so that the rank ℓ sums in (33) are diagonal. If $\ell > 1$, there exist constants $M_{n,\ell}$ and $N_{n,\ell}$ such that

$$\#R \leq M_{n,\ell} \quad \textit{and} \quad \#T \leq N_{n,\ell}.$$

Proof. Write $p(S) = (p_1, \ldots, p_s)$, where $\ell \leq s \leq n$. By the proof of Theorem 4.3, we know how to pass from a sum of type

$$(p_1, \ldots, p_i, \ldots, p_\ell, \ldots, p_s) \tag{34}$$

to a linear combination of sums of types

$$(p_1, \ldots, p_i - 1, \ldots, p_\ell, \ldots, p_s + 1), \quad i = 1, \ldots, \ell. \tag{35}$$

By iterating this, we pass from S to a linear combination of sums with types

$$(p'_1, \ldots, p'_{i-1}, 0, p'_{i+1}, \ldots, p'_\ell, \ldots, p'_s), \quad i = 1, \ldots, \ell. \tag{36}$$

We will bound the number of rank ℓ (respectively rank $\ell - 1$) sums produced in passing from (34) to (36) by a constant $M^{(s)}_{n,\ell}$ (resp. $N^{(s)}_{n,\ell}$). We can then take

$$M_{n,\ell} = \prod_{s=\ell+1}^{n} M^{(s)}_{n,\ell},$$

and similarly for $N_{n,\ell}$.

To describe what happens in going from (34) to (36), we use a geometric construction. Let $B = B(p_1, \ldots, p_\ell)$ be the set

$$B = \left\{(x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell \;\middle|\; 0 \leq x_i \leq p_i, i = 1, \ldots, \ell\right\}.$$

The points in B correspond to types of intermediate sums in the passage from (34) to (36). In particular, passing from (34) to (35) can be encoded by moving from $(x_1, \ldots, x_\ell)$ to $(x_1, \ldots, x_i - 1, \ldots, x_\ell)$ in B. Moreover, the sums of the form (36) correspond to the subset of points $B_0 \subset B$ with exactly one coordinate 0. (See Figure 1.)

Now the constant $M^{(s)}_{n,\ell}$ will be given by $\operatorname{Max} \#B_0$, as $B(p_1, \ldots, p_\ell)$ ranges over all possibilities for fixed n, ℓ, and s. If we allow the p_i to become continuous parameters, then a simple computation shows that the maximum occurs when $p_1 = \cdots = p_\ell = (n - s + \ell)/\ell$. With these conditions we have

$$M^{(s)}_{n,\ell} \leq \ell \left(\frac{n - s + \ell}{\ell}\right)^{\ell - 1}. \tag{37}$$

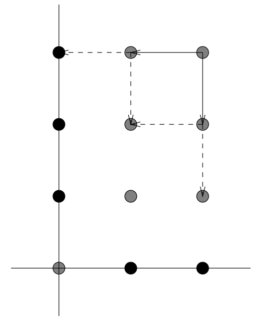

FIGURE 1. The set $B(2,3)$. The arrows represent applications of Proposition 7. The black dots represent points in B_0.

The constant $N^{(s)}_{n,\ell}$ can be computed similarly. There are $(\ell+1)$ sums of rank $\ell-1$ produced for each point in

$$B_+ := \bigl\{(x_1,\dots,x_\ell)\in B \bigm| x_i \neq 0\bigr\}.$$

One finds again that the maximum occurs when $p_1 = \cdots = p_\ell = (n-s+\ell)/\ell$, and is

$$N^{(s)}_{n,\ell} \leq (\ell+1)\left(\frac{n-s+\ell}{\ell}\right)^{\ell}.$$

□

5.3. Proposition. *If n and ℓ are fixed, then* (33) *can be constructed in constant time, independent of $\|S\|$.*

Proof. This follows easily from the proof of Lemma 5.2. Forming the expression (33) is purely combinatorial, and makes no reference to $\|S\|$. In particular, the number of steps needed can be bounded for fixed n and ℓ. □

5.4. Now we investigate the size of the output in Theorem 4.5. The main step of the proof of Theorem 4.5 shows how given S, one may write

$$S = \sum_{\varrho\in R} q_\varrho S(Z^\ell,\varrho) + \sum_{\tau\in T} q_\tau S(Z^{\ell-1},\tau), \tag{38}$$

where the sums of rank ℓ on the right satisfy $\|S(Z^\ell,\varrho)\| < \|S\|^{(n-1)/n}$. Denote by C_ℓ (respectively $C_{\ell-1}$) the number of rank ℓ (resp. rank $\ell-1$) sums on the right of (38).

5.5. Lemma. *We have $C_\ell \leq M^{(\ell+1)}_{n+1,\ell}$ and $C_{\ell-1} \leq N^{(\ell+1)}_{n+1,\ell}$.*

Proof. The proof is very similar to the that of Lemma 5.2, with the following twist: we begin with a diagonal sum, increase the number of distinct linear forms by one, and then make the sums diagonal again. We can keep track of the number of sums

produced using the set B as in the preceding proof, although we must replace n with $n+1$ to accommodate the extra initial step. We leave the details to the reader. $\square$

5.6. Now consider the expression (38). To complete the proof of Theorem 4.5, we repeat the process that produced (38) until all rank ℓ sums are diagonal and unimodular, and we obtain the expression (32). Using Lemma 5.5 and the estimate (6), we can bound the number of rank ℓ sums produced.

5.7. Proposition. *(cf.* [2, Theorem 5.4]*) Write S as a sum of diagonal, unimodular rank ℓ sums and lower rank sums as in* (32)*. Then the number of rank ℓ sums on the right of* (32) *is bounded by*

$$C'(\log\|S\|)^{A\log C_\ell}, \tag{39}$$

where $A=\log(n/(n-1))^{-1}$, and C' is a constant independent of S.

Proof. In (38), we have

$$0\le\|S(Z^\ell,\varrho)\|<\|S\|^{(n-1)/n}.$$

Thus after t iterations we'll have

$$0\le\|S(Z^\ell,\varrho)\|<\|S\|^{((n-1)/n)^t}.$$

Since the index of a Dedekind sum is always an integer, the condition for termination is that for some $\varepsilon>0$, we have

$$\|S\|^{((n-1)/n)^t}\le 2-\varepsilon,\quad\text{or}\quad t\ge\frac{\log\log\|S\|-\log\log(2-\varepsilon)}{\log n-\log(n-1)}. \tag{40}$$

On the other hand, by Lemma 5.5, we know that t iterations will produce no more than C_ℓ^t sums of rank ℓ. So fix $\varepsilon>0$, set $A=\log n-\log(n-1)$, and let

$$C'_\ell=\exp\left(\left(\frac{-\log\log(2-\varepsilon)}{A}+1\right)\log C_\ell\right).$$

Then

$$C_\ell^t\le C'_\ell\big(\log\|S\|\big)^{A\log C_\ell}.$$

Now if we define $C'=\mathrm{Max}_{\ell\le n}C'_\ell$, we obtain (39). $\square$

5.8. We are now ready to discuss the complexity of our algorithms.

5.9. Theorem. *Let $S=S(Z^\ell,\sigma)$ be a normalized, properly embedded Dedekind sum. Using Corollary 4.6, write S as a $\mathbb{Q}$-linear combination of diagonal unimodular sums. Then there exists a polynomial $P_{n,\ell}$ such that the number of terms in the expression is bounded by $P_{n,\ell}(\log\|S\|)$. Moreover, we have*

$$\deg P_{n,\ell}\le A\log\left(\frac{\ell!\,n^{\ell(\ell-1)/2}}{2^1 3^2\cdots\ell^{(\ell-1)}}\right),$$

where $A=\log(n/(n-1))^{-1}$.

Proof. Fix n. We proceed by induction on ℓ.

First, if $\ell = 1$, then by Lemma 3.9 the sum $S(Z^1, \sigma)$ is already diagonal and unimodular. Hence we may take $P_{n,1} \equiv 1$.

Next, assume that the statement is true for sums of rank $\ell - 1$, and let $P_{n,\ell-1}$ be the corresponding polynomial. First we claim that without loss of generality, we need only consider the case that S is diagonal. Indeed, apply Theorem 4.3 and write

$$S = \sum_{\varrho \in R} q_\varrho S(Z^\ell, \varrho) + \sum_{\tau \in T} q_\tau S(Z^{\ell-1}, \tau), \tag{41}$$

where the rank ℓ Dedekind sums are diagonal, and all the Dedekind sums have index $\leq \|S\|$. By Lemma 5.2, the sets R and T have a bounded number of elements independent of S. Hence we may bound the output for diagonal S, and then multiply this by a constant to obtain our final answer.

Now we apply Theorem 4.5, and we must count the number of rank Dedekind sums produced. By Proposition 5.7, we know that the total number of rank ℓ sums will be bounded by

$$C'(\log \|S\|)^{A \log C_\ell}. \tag{42}$$

Furthermore, each sum of lower rank produced in the proof of Theorem 4.5 can be written as a sum of $\leq P_{n,\ell-1}(\|S\|)$ Dedekind sums by induction. To find the total output, we must count these lower rank sums.

We can do this as follows. Let $Q = Q_{n,\ell} := C_{\ell-1} P_{n,\ell-1}$, where $C_{\ell-1}$ is the constant in Lemma 5.5. Represent the process of reducing the diagonal sum S to unimodularity by the following diagram:

$$\begin{tikzcd}
\text{rank } \ell\text{:} & 1 \arrow[r] \arrow[dr] & C_\ell \arrow[r] \arrow[dr] & C_\ell^2 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & C_\ell^t \\
\text{rank } \ell - 1\text{:} & & Q & QC_\ell & \cdots & QC_\ell^{t-1}
\end{tikzcd} \tag{43}$$

The top row represents the bound on the number of rank ℓ sums at each step of the algorithm, and the bottom row is the number of rank $\ell - 1$ sums.

According to (43), we have produced

$$Q \sum_{i=0}^{t-1} C_\ell^i = Q \, \frac{C_\ell^t - 1}{C_\ell - 1} \leq Q \, \frac{C'(\log \|S\|)^{A \log C_\ell} - 1}{C_\ell - 1}$$

sums of ranks $< \ell$. Adding this estimate to (42), we find that the total number of Dedekind sums produced will be a polynomial of degree

$$\deg Q + A \log C_\ell = \deg P_{n,\ell-1} + A \log C_\ell.$$

To complete the proof, we compute the degree of $P_{n,\ell}$ by induction. Indeed, using the estimate

$$C_\ell \le M^{(\ell+1)}_{n+1,\ell} = \ell \left(\frac{n}{\ell}\right)^{\ell-1},$$

and that $\deg P_{n,1} = 0$, an easy computation shows that

$$\deg P_{n,\ell} \le A \log \left(\frac{\ell!\, n^{\ell(\ell-1)/2}}{2^1 3^2 \cdots \ell^{(\ell-1)}} \right),$$

as required. ☐

5.10. Example. Here is a table of the bound of $\deg P_{n,\ell}$ for small values of n and ℓ.

n,ℓ	2	3	4	5	6	7	8
2	1						
3	2	5					
4	4	10	15				
5	7	16	25	33			
6	9	23	37	50	60		
7	12	30	50	69	86	99	
8	15	38	64	90	114	135	150

5.11. Corollary. *Keeping the same notation as in Theorem 5.9, for fixed n and ℓ we may express S as a finite $\mathbb{Q}$-linear combination of diagonal unimodular sums in time polynomial in* $\log \|S\|$.

Proof. First, the vector w constructed in Proposition 2.2 can be found in polynomial time in the size of the coefficients of S. In fact, investigation of [2, Lemma 5.2] shows that the rational numbers α_j in (29) in the proof of Theorem 4.3 can also be constructed in polynomial time. This implies that w and the α_j can be found in time polynomial in $\log \|S\|$. The proof of Theorem 5.9 then shows that the final expression can be computed in polynomial time. ☐

6. The Eisenstein cocycle

6.1. In this section, we briefly review the construction of the Eisenstein cocycle introduced in [7]. In particular, we show that this is a finite object that can be calculated effectively using Corollary 4.6. Roughly speaking, the Eisenstein cocycle represents a generalization of the classical Bernoulli polynomial within the arithmetic of the unimodular group $\Gamma = GL_n(\mathbb{Z})$.

6.2. Let $\mathscr{A} = (A_1, \dots, A_n)$ be an n-tuple of matrices $A_i \in GL_n(\mathbb{R})$. For an n-tuple $d = (d_1, \dots, d_n)$ of integers $1 \le d_i \le n$, let $\mathscr{A}(d) \subseteq \mathbb{R}^n$ be the subspace generated by all columns A_{ij} such that $j < d_i$. (Here A_{ij} denotes the jth column of A_i.) Writing $\mathscr{A}(d)^\perp$ for the orthogonal complement of $\mathscr{A}(d)$ in $\mathbb{R}^n$, we let

$$X(d) = \mathscr{A}(d)^\perp \smallsetminus \bigcup_{i=1}^{n} \sigma_i^\perp, \quad \text{where } \sigma_i = A_{id_i}. \tag{44}$$

The n-tuple $\mathscr{A}$ determines then the stratification

$$\mathbb{R}^n \smallsetminus \{0\} = \bigsqcup_{d \in D} X(d), \tag{45}$$

indexed by the finite set

$$D = D(\mathscr{A}) = \{d \mid X(d) \neq \varnothing\}.$$

Associated to this decomposition is the rational function $\psi(\mathscr{A})$ on $\mathbb{R}^n \smallsetminus \{0\}$ defined by

$$\psi(\mathscr{A})(x) = \frac{\det(\sigma_1, \dots, \sigma_n)}{\langle x, \sigma_1 \rangle \cdots \langle x, \sigma_n \rangle}, \quad \text{if } x \in X(d).$$

6.3. More generally, if $P(X_1, \dots, X_n)$ is any homogeneous polynomial, we form the differential operator $P(-\partial_{x_1}, \dots, -\partial_{x_n})$ in the partial derivatives $\partial_{x_i} := \partial/\partial x_i$, and define

$$\psi(\mathscr{A})(P, x) = P(-\partial_{x_1}, \dots, -\partial_{x_n})\psi(\mathscr{A})(x).$$

The last expression can be written more explicitly as

$$\psi(\mathscr{A})(P, x) = \det(\sigma) \sum_r P_r(\sigma) \prod_{j=1}^{n} \frac{1}{\langle x, \sigma_j \rangle^{1+r_j}}, \tag{46}$$

where r runs over all decompositions of $\deg(P) = r_1 + \cdots + r_n$ into nonnegative parts $r_j \ge 0$, and $P_r(\sigma)$ is the homogeneous polynomial in the σ_{ij} defined by the expansion

$$P(X\sigma^t) = \sum_r P_r(\sigma) \prod_{j=1}^{n} \frac{X_j^{r_j}}{r_j!}.$$

In the excluded case $x = 0$, it is convenient to set $\psi(\mathscr{A})(P, 0) = 0$.

The definition of the Eisenstein cocycle Ψ is now easy to state:

$$\Psi(\mathscr{A})(P, Q, v) := (2\pi i)^{-n-\deg P} \sum_{x \in \mathbb{Z}^n} \mathbf{e}(\langle x, v \rangle) \psi(\mathscr{A})(P, x) \Big|_Q. \tag{47}$$

The series on right converges provided all components A_i of $\mathscr{A}$ are in $GL_n(\mathbb{Q})$. However, since the convergence is only conditional, we are forced to introduce the additional parameter Q specifying the limiting process.

6.4. Let M be the set of all complex-valued functions $f(P,Q,v)$ with P, Q, v as above ($v \in \mathbb{R}^n$). Then M is a left Γ-module under the action

$$Af(P,Q,v) = \det(A) f(A^t P, A^{-1}Q, A^{-1}v), \quad A \in \Gamma,$$

where the implied Γ-action on homogeneous polynomials is given by $(AP)(X) = P(XA)$. With respect to this action, the map $\Psi\colon \Gamma^n \to M$ has the property

$$\Psi(A\mathscr{A}) = A\Psi(\mathscr{A}), \quad A \in \Gamma,\ \mathscr{A} \in \Gamma^n, \tag{48}$$

$$\sum_{i=0}^{n} \Psi(A_0, \ldots, \widehat{A}_i, \ldots, A_n) = 0, \quad A_i \in \Gamma. \tag{49}$$

In other words, Ψ is a homogeneous cocycle on Γ. It is known that Ψ represents a nontrivial cohomology class in $H^{n-1}(\Gamma; M)$ [7, Theorem 4].

Combining (44)–(47), we see that Ψ is a finite linear combination of Dedekind sums,

$$\Psi(\mathscr{A})(P,Q,v) = (2\pi i)^{-n-\deg P} \sum_{d\in D} \sum_{r} P_r(\sigma) S(L,\sigma,e,v)\Big|_Q.$$

Here σ is the matrix with columns A_{id_i} for $i = 1, \ldots, n$, L is the lattice $\mathscr{A}(d)^\perp \cap \mathbb{Z}^n$, and $e_j = 1 + r_j$. The case $P = 1$ is of special interest:

$$\Psi(\mathscr{A})(1,Q,v) = (2\pi i)^{-n} \sum_{d\in D} S(L,\sigma,\mathbf{1},v)\Big|_Q.$$

This case yields the classical Dedekind-Rademacher sums if $n = 2$, and, more importantly, it corresponds to special values of partial zeta functions at $s = 0$.

7. Values of partial zeta functions

7.1. Let F be a totally real number field of degree n over $\mathbb{Q}$, and let $\mathfrak{f}$, $\mathfrak{b}$ be two relatively prime ideals in the ring of integers $\mathscr{O}_F$. The partial zeta function to the ray class $\mathfrak{b} \mod \mathfrak{f}$ is defined by

$$\zeta(\mathfrak{b},\mathfrak{f},s) := \sum_{\mathfrak{a}\equiv\mathfrak{b} \mod \mathfrak{f}} N(\mathfrak{a})^{-s}, \quad \Re(s) > 1,$$

where $\mathfrak{a}$ runs over all all integral ideals in $\mathscr{O}_F$ such that the fractional ideal $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal generated by a totally positive number in the coset $1 + \mathfrak{f}\mathfrak{b}^{-1}$. According to Klingen-Siegel, the special values $\zeta(\mathfrak{b},\mathfrak{f},1-s)$, where $s = 1, 2, 3, \ldots$, are well-defined rational numbers. In this section, we give a formula for calculating these numbers in terms of the Eisenstein cocycle Ψ.

7.2. The formula depends on the choice of a $\mathbb{Z}$-basis W for the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1} = \sum \mathbb{Z}W_j$, together with the dual basis W^* determined by $\mathrm{Tr}(W_i^* W_j) = \delta_{ij}$. Here we identify $\alpha \in F$ with the row vector $(\alpha^{(1)}, \dots, \alpha^{(n)}) \in \mathbb{R}^n$, where the $\alpha^{(j)}$ are the n different embeddings of α into the field of real numbers. Then W can be identified with a matrix in $GL_n(\mathbb{R})$ whose jth row is the basis vector W_j. Let

$$P(X) = N(\mathfrak{b}) \prod_i \sum_j X_j W_j^{(i)},$$
$$Q(X) = \prod_i \sum_j X_j (W_j^*)^{(i)},$$

and let $v \in \mathbb{Q}^n$ be defined by $v_j = \mathrm{Tr}(W_j^*)$.

The formula also depends on the choice of generators $\varepsilon_1, \dots, \varepsilon_\nu$, where $\nu = n - 1$, for the group $U \subset \mathscr{O}_F^\times$ of totally positive units. Using the regular representation $\rho\colon U \to \Gamma$, defined via $\rho(\varepsilon) = W\delta(\varepsilon)W^{-1}$, where $\delta(\varepsilon)$ is the matrix $\mathrm{diag}(\varepsilon^{(1)}, \dots, \varepsilon^{(n)})$, we identify the units ε_j with elements $A_j = \rho(\varepsilon_j)^t \in \Gamma$. (Note that ρ is the *row* regular representation.)

7.3. Using the bar notation

$$[A_1|\cdots|A_\nu] := (1, A_1, A_1A_2, \dots, A_1 \cdots A_\nu) \in \Gamma^n,$$

we have the following proposition expressing the zeta values in terms of the Eisenstein cocycle:

7.4. Proposition. *Let $U_\mathfrak{f}$ be the subgroup $U \cap (1 + \mathfrak{f})$, and let π run through all permutations of $\{1, \dots, \nu\}$. Then for $s = 1, 2, 3, \dots,$*

$$\zeta(\mathfrak{b}, \mathfrak{f}, 1 - s) = \eta \sum_{\varepsilon \in U/U_\mathfrak{f}} \sum_\pi \mathrm{sgn}(\pi) \Psi([A_{\pi(1)}|\cdots|A_{\pi(\nu)}])(P^{s-1}, Q, \rho(\varepsilon)^t v).$$

Here the sign $\eta = \pm 1$ is determined by

$$\eta = (-1)^\nu \,\mathrm{sgn}(\det W) \,\mathrm{sgn}(R),$$

where $R = \det(\log \varepsilon_i^{(j)})$, $1 \le i, j \le \nu$.

Proof. This follows from [7, Corollary, p. 595] by writing the fundamental cycle of $U_\mathfrak{f}$ in terms of the A_j. □

7.5. Example. We work out the above formula in the case of a real quadratic field F. Let $\varepsilon > 1$ be the fundamental unit of U, the group of totally positive units in F, and let $\mathbb{Z}w_1 + \mathbb{Z}w_2 = \mathfrak{f}\mathfrak{b}^{-1}$ be a $\mathbb{Z}$-basis of $\mathfrak{f}\mathfrak{b}^{-1}$. Such a basis determines a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and a vector $v \in \mathbb{Q}^2$ via

$$\begin{pmatrix} \varepsilon w_1 \\ \varepsilon w_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad v_1 w_1 + v_2 w_2 = 1.$$

In addition, we get the normforms

$$P(X) = N(x_1w_1 + x_2w_2), \quad Q(X) = N(x_1w_2 - x_2w_1).$$

Let p be the smallest positive integer such that $(A^p - 1)v \in \mathbb{Z}^2$. Then

$$\zeta_F(\mathfrak{b}, \mathfrak{f}, 1-s) = \eta \sum_{k \mod p} \Psi(1, A)(P^{s-1}, Q, A^k v), \tag{50}$$

where $\eta = \mathrm{sgn}(w_2 w_1^{(1)} - w_1 w_2^{(1)})$, and, if $s = 1$,

$$\Psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(1, Q, v) = \frac{a}{2c}\mathscr{B}_2(v_2) + \frac{d}{2c}\mathscr{B}_2(cv_1 - av_2) \tag{51}$$

$$- \sum_{j \mod c} \mathscr{B}_1\left(\frac{j+v_2}{|c|}\right)\mathscr{B}_1\left(a\frac{j+v_2}{c} - v_1\right), \tag{52}$$

with an additional correction term $-\mathrm{sgn}(c)/4$ on the right if $v \in \mathbb{Z}^2$. The finite sum (52) is the classical Dedekind-Rademacher sum $(2\pi i)^{-2} S\left(\mathbb{Z}^2, \left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right), \mathbf{1}, v\right)\big|_Q$. Note that the number of terms in that sum equals $|c| = |(\varepsilon - \varepsilon')/(w - w')|$, where $w = w_2/w_1$, and the prime is Galois conjugation. Depending on ε, this number can be very large. To get a more efficient formula for calculating Ψ, we apply the euclidean algorithm to the first column of A an obtain a product decomposition

$$A = B_1 \cdots B_t, \quad t \geq 1, \quad B_j = \begin{pmatrix} b_j & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\Psi(1, A) = \sum_{\ell=0}^{t-1} (B_1 \cdots B_\ell)\Psi(1, B_{\ell+1}). \tag{53}$$

Here t is roughly $\log|c|$. Thus the number of terms is effectively reduced from $|c|$ to $\log|c|$, since

$$\begin{aligned} &\Psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}\right)(P^{s-1}, Q, v) \\ &= \sum_r \left[bP_r\begin{pmatrix} 0 & b \\ 1 & 1 \end{pmatrix} \frac{\mathscr{B}_{2s}(v_2)}{(2s)!} + P_r\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \frac{\mathscr{B}_{1+r_1}(v_1 - bv_2)}{(1+r_1)!}\frac{\mathscr{B}_{1+r_2}(v_2)}{(1+r_2)!}\right], \end{aligned} \tag{54}$$

where the rational numbers $P_r\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ are the coefficients of the polynomial

$$P(\alpha X_1 + \beta X_2, \gamma X_1 + \delta X_2)^{s-1} = \sum_r P_r\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \frac{X_1^{r_1}}{r_1!}\frac{X_2^{r_2}}{r_2!}.$$

In the exceptional case $s = 1$, $\mathfrak{f} = (1)$, the correction term

$$-\frac{1}{8}\{\mathrm{sgn}(w + b) + \mathrm{sgn}(w' + b)\}$$

must be added to the right side of (54).

As a numerical example, we choose $F = \mathbb{Q}(\sqrt{5})$, $\varepsilon = (3+\sqrt{5})/2$, $\mathfrak{f} = \mathfrak{b} = (1)$, $w_1 = -\varepsilon$, $w_2 = 1$. Then $\eta = +1$, $A = \left(\begin{smallmatrix} 3 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $P_{11}\left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right) = 3$, while $P_{20}\left(\begin{smallmatrix} 0 & 3 \\ 1 & 1 \end{smallmatrix}\right) = 2$, $P_{11}\left(\begin{smallmatrix} 0 & 3 \\ 1 & 1 \end{smallmatrix}\right) = -7$, $P_{02}\left(\begin{smallmatrix} 0 & 3 \\ 1 & 1 \end{smallmatrix}\right) = 2$. Hence, according to (50) and (54), we get for the value of the Dedekind zeta function of F at $s = -1$,

$$\zeta_F(-1) = \zeta((1),(1),-1) = 3(2-7+2)(-\frac{1}{720}) + 3(\frac{1}{12})^2 = \frac{1}{30}.$$

7.6. Example. As a second example, we consider the cubic field $\mathbb{Q}(\theta)$ of discriminant 148 given by $\theta^3 - \theta^2 - 3\theta + 1 = 0$. According to [6], the group of totally positive units U is generated by $\varepsilon_1 = -3\theta^2 + 2\theta + 10$ and $\varepsilon_2 = 5\theta^2 + 6\theta - 2$.

Let $\mathfrak{f} = (2)$ and $\mathfrak{b} = (1)$. Then $\mathfrak{f}\mathfrak{b}^{-1} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3$, where $w_1 = 2$, $w_2 = 2\theta$, and $w_3 = 2\theta^2$. With respect to this basis, we find

$$A_1 = \begin{pmatrix} 10 & 3 & 1 \\ 2 & 1 & 0 \\ -3 & -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -2 & -5 & -11 \\ 6 & 13 & 28 \\ 5 & 11 & 24 \end{pmatrix}, A_1A_2 = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 3 & 6 \\ 0 & 2 & 5 \end{pmatrix}.$$

Then $v = (1/2, 0, 0)^t$ and $\eta = 1$. Let $V \subset \mathbb{Q}^3$ be a complete set of representatives for the orbit of $v + \mathbb{Z}^3$ under the action of U (via A_1 and A_2) on $\mathbb{Q}^3/\mathbb{Z}^3$. Note that V is a finite set. Thus

$$\zeta(\mathfrak{b},\mathfrak{f},0) = \sum_{v\in V} \big(\Psi(1, A_1, A_1A_2) - \Psi(1, A_2, A_1A_2)\big)(1, Q, v). \tag{55}$$

Each term on the right of (55) breaks up into 10 Dedekind sums: one of rank 3, three of rank 2, and six of rank 1. Note that the rank 3 and rank 1 sums are diagonal, whereas the rank 2 sums are not. After making all sums diagonal, we find that each Ψ in (55) has 30 terms. Applying the summation formula for diagonal sums (Proposition 3.11), we see that to evaluate Ψ on any element of V, we must sum 76 terms.

Since $\varepsilon_2^2 \equiv \varepsilon_1\varepsilon_2 \equiv 1 \mod \mathfrak{f}$, we can take $V = \{v, A_2v\}$. This yields 152 terms altogether, all of which sum to 0, and thus $\zeta((1),(2),0) = 0$. This agrees with [6], and also with the observation that since -1 preserves the congruence class of $1+\mathfrak{f}$, and the norm of -1 is -1, the special value at $s = 0$ must vanish.

Now let $\mathfrak{f} = (3)$. Then we may take A_1, A_2 as above, and $v = (1/3, 0, 0)^t$ and $\eta = 1$. Since $\varepsilon_2^{13} \equiv \varepsilon_1\varepsilon_2^5 \equiv 1 \mod \mathfrak{f}$, we must sum $13 \cdot 76 = 988$ terms, and we find $\zeta((1),(3),0) = 2/3$, again in agreement with [6].

Note that to compute $\zeta((1),(N),0)$ for various $N \in \mathbb{Z}$, we must only compute A_1, A_2, and thus $\Psi(1, A_1, A_1A_2) - \Psi(1, A_2, A_1A_2)$ once. After this it is routine to compute special values at $s = 0$, and the complexity in (55) comes from $\#V$, which can be large, even for small values of N. A table of $\zeta((1),(N),0)$ for several rational integers N is given below. The values for $N = 2, 3, 5, 7$ are also in [6].

N	$N\cdot\zeta$	N	$N\cdot\zeta$	N	$N\cdot\zeta$	N	$N\cdot\zeta$	N	$N\cdot\zeta$
		11	-18	21	-78	31	74	41	382
2	0	12	5	22	68	32	15	42	-228
3	2	13	-22	23	12	33	62	43	-366
4	1	14	-20	24	23	34	50	44	1
5	-4	15	42	25	106	35	-54	45	254
6	4	16	7	26	-24	36	-43	46	6
7	2	17	100	27	-190	37	20	47	-570
8	3	18	-32	28	25	38	22	48	-13
9	-10	19	82	29	242	39	156	49	-222
10	-2	20	4	30	6	40	2	50	178

$\zeta((1),(N),0)$ for the cubic field of discriminant 148.

8. Witten's zeta function

8.1. To give another illustration, we recall the definition of Witten's zeta function [8], and show how our algorithms can be used to compute special values of this function at even integers. For unexplained notions from representation theory, the reader may consult [4].

Let $\mathfrak{g}$ be a simple complex lie algebra, and let R be the associated root system. Let R^+ (respectively R^-) be a subset of positive roots (resp. negative roots), and let $\Delta \subset R^+$ be the set of simple roots.

The roots R generate a lattice Λ_R in an ℓ-dimensional real vector space E endowed with an inner product $(\ ,\ )$. Let Λ_W be the weight lattice, which is the dual of Λ_R with respect to this inner product. We denote by $\Omega \subset E$ the set of fundamental weights, which is the basis of Λ_W dual to Δ.

Let W be the Weyl group of R. This is a finite group that acts on E via a reflection representation, and preserves the inner product and the lattices Λ_R and Λ_W. There is a decomposition of E into a finite union of rational polyhedral cones, and W acts by permuting these cones. Let C^+ be the closed top-dimensional cone generated by Ω.

Let Π denote the set of isomorphism classes of complex irreducible representations of $\mathfrak{g}$. It is known that elements of Π are in bijection with the set $\Lambda_W \cap C^+$, the dominant weights. Given λ from this latter set, we denote the corresponding representation by π_λ. Then the definition of the zeta function associated to $\mathfrak{g}$ is

$$\zeta_{\mathfrak{g}}(s) := \sum_{\lambda\in\Lambda_W\cap C^+} (\dim \pi_\lambda)^{-s}. \tag{56}$$

8.2. Let $m > 1$ be an integer. The special value $\zeta_{\mathfrak{g}}(2m)$ can be computed using a Dedekind sum as follows.

Let ρ be one-half the sum of the positive roots. An application of the Weyl character formula [4, Corollary 24.6] shows that for any dominant weight λ, we have

$$\dim \pi_\lambda = \prod_{\alpha\in R^+} \frac{(\rho+\lambda,\alpha)}{(\rho,\alpha)}. \tag{57}$$

It is known that any dominant weight λ can be written as a nonnegative integral linear combination of the fundamental weights. Using this in (57), a computation shows that (56) becomes

$$\zeta_{\mathfrak{g}}(2m) = M^{2m} \sum_{x\in(\mathbb{Z}^{>0})^\ell} \frac{1}{\prod_{i=1}^r \langle a_i, x\rangle^{2m}}. \tag{58}$$

Here M is the integer $\prod_{\alpha\in R^+}(\rho,\alpha)$, r is the number of positive roots, and the $a_i \in (\mathbb{Z}^{>0})^\ell$ are the coefficients of the positive roots in terms of Δ. The pairing $\langle\ ,\ \rangle$ is the usual scalar product on $\mathbb{R}^\ell$.

8.3. We obtain a Dedekind sum by extending the sum (58) to the whole lattice. Let $Z^\ell \subset \mathbb{R}^r$ be the span of the first ℓ basis vectors, and let $\sigma = \sigma(\mathfrak{g})$ be an $r\times r$ integral matrix such that $\langle \sigma_i, x\rangle = \langle a_i, x\rangle$ for $x \in Z^\ell$, and such that $\det\sigma = 1$. Let $e = (2k,\dots,2k) \in \mathbb{R}^r$.

8.4. Proposition. *(cf.* [9, p. 507]*)* $\zeta_{\mathfrak{g}}(2k) = \frac{M^{2k}}{\#W} S(Z^\ell, \sigma(\mathfrak{g}), e, 0)$.

Hence these special values can be computed in polynomial time using our techniques. We conclude with two examples: $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$. We recommend verification of these formulas to the interested reader for a pleasant combinatorial exercise.

8.5. Proposition. *Let $\zeta(s)$ be the Riemann zeta function. Then*

$$\frac{6}{2^{2m}}\zeta_{\mathfrak{sl}_3}(2m) = 8\sum_{\substack{0\le i\le 2m\\ i\equiv 0 \mod 2}}\binom{4m-i-1}{2m-1}\zeta(i)\zeta(6m-i). \tag{59}$$

$$\frac{24}{12^{2m}}\zeta_{\mathfrak{sl}_4}(2m) = 16\sum_{0\le i\le 2m}\binom{4m-i-1}{2m-1}(A+B+C+D), \quad \textit{where} \tag{60}$$

(61)
$$A=\sum_{\substack{0\le j\le 2m\\ 0\le t\le 4m+i-j\\ j,t\equiv 0 \mod 2}}\binom{2m+i-j-1}{i-1}\binom{6m+i-j-t-1}{2m-1}\zeta(j)\zeta(t)\zeta(12m-j-t),$$

(62)
$$B=\sum_{\substack{0\le j\le 2m\\ 0\le u\le 2m\\ j,u\equiv 0 \mod 2}}\binom{2m+i-j-1}{i-1}\binom{6m+i-j-u-1}{4m+i-j-1}\zeta(j)\zeta(u)\zeta(12m-j-u),$$

(63)
$$C=\sum_{\substack{0\le k\le i\\ 0\le v\le 4m+i-k\\ k,v\equiv 0 \mod 2}}\binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}\zeta(k)\zeta(v)\zeta(12m-k-v),$$

(64)
$$D=\sum_{\substack{0\le k\le i\\ 0\le w\le 2m\\ k,w\equiv 0 \mod 2}}\binom{2m+i-k-1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1}\zeta(k)\zeta(w)\zeta(12m-k-w).$$

8.6. *Remark.* The formula (59) was independently discovered by Zagier, S. Garoufalidis, and L. Weinstein [9, p. 506].

8.7. Example. Here are some special values of $\zeta_{\mathfrak{sl}_3}$ and $\zeta_{\mathfrak{sl}_4}$.

$2m$	$(6m+1)!\cdot 6\cdot \zeta_{\mathfrak{sl}_3}(2m)/(2^{2m}\cdot(2\pi)^{6m})$
2	$1/(2\cdot 3)$
4	$19/(2\cdot 3\cdot 5)$
6	$1031/(3\cdot 7)$
8	$(11\cdot 43\cdot 751)/(2\cdot 7)$
10	$(5\cdot 13\cdot 27739097)/(3\cdot 11)$
12	$(17\cdot 29835840687589)/(3\cdot 5\cdot 7\cdot 13)$
14	$(2\cdot 17\cdot 19\cdot 89\cdot 127\cdot 6353243297)/7$
16	$(19\cdot 23\cdot 31\cdot 221137132669842886663)/(2\cdot 5^2\cdot 13\cdot 17)$

$2m$	$(12m+1)! \cdot (6m+1) \cdot (4m+1) \cdot 24 \cdot \zeta_{\mathfrak{sl}_4}(2m)/(12^{2m} \cdot (2\pi)^{12m})$
2	$23/2$
4	$(3 \cdot 7 \cdot 14081)/2$
6	$(757409 \cdot 23283173)/(5 \cdot 7)$
8	$(3 \cdot 11 \cdot 1021 \cdot 5529809 \cdot 754075957)/2$
10	$(13 \cdot 116763209 \cdot 1872391681 \cdot 3187203549787)/(5 \cdot 11)$
12	$(17 \cdot 1798397149 \cdot 5509496891 \cdot 6127205846988571484743)/(3 \cdot 7 \cdot 13)$

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